(Spline, Bezier, B-Spline)
Spline

• Drafting terminology
  – Spline is a flexible strip that is easily flexed to pass through a series of design points (control points) to produce a smooth curve.

• Spline curve – a piecewise polynomial (cubic) curve whose first and second derivatives are continuous across the various curve sections.
Beziers curve

- Developed by Paul de Casteljau (1959) and independently by Pierre Bezier (1962).
- French automobil company – Citroen & Renault.
Parametric function

- \( P(u) = \sum_{i=0}^{n} B_{n,i}(u)p_i \)

Where

\[
B_{n,i}(u) = \frac{n!}{i!(n-i)!} u^i (1-u)^{n-i} \quad 0 \leq u \leq 1
\]

For 3 control points, \( n = 2 \)
\[ P(u) = (1-u)^2p_0 + 2u(1-u)p_1 + u^2p_2 \]

For four control points, \( n = 3 \)
\[ P(u) = (1-u)^3p_0 + 3u(1-u)^2p_1 + 3u^2(1-u)p_2 + u^3p_3 \]
algorithm

• De Casteljau
  – Basic concept
  • To choose a point C in line segment AB such that C divides the line segment AB in a ratio of \( u : 1-u \)

Let \( u = 0.5 \)
  \( u=0.25 \)
  \( u=0.75 \)
properties

- The curve passes through the first, $P_0$ and last vertex points, $P_n$.
- The tangent vector at the starting point $P_0$ must be given by $P_1 - P_0$ and the tangent $P_n$ given by $P_n - P_{n-1}$
- This requirement is generalized for higher derivatives at the curve’s end points. E.g 2nd derivative at $P_0$ can be determined by $P_0, P_1, P_2$ (to satisfy continuity)
- The same curve is generated when the order of the control points is reversed
Properties (continued)

• Convex hull
  – Convex polygon formed by connecting the control points of the curve.
  – Curve resides completely inside its convex hull
B-Spline

• Motivation (recall bezier curve)
  – The degree of a Bezier Curve is determined by the number of control points
  – E. g. (bezier curve degree 11) – difficult to bend the "neck" toward the line segment \( P_4 P_5 \).
  – Of course, we can add more control points.
  – BUT this will increase the degree of the curve \( \rightarrow \) increase computational burden
B-Spline

• Motivation (recall bezier curve)
  – Joint many bezier curves of lower degree together (right figure)
  – BUT maintaining continuity in the derivatives of the desired order at the connection point is not easy or may be tedious and undesirable.
B-Spline

• Motivation (recall bezier curve)
  – moving a control point affects the shape of the entire curve- *(global modification property)* – undesirable.
  - Thus, the solution is B-Spline – the degree of the curve is independent of the number of control points
  - E.g - right figure – a B-spline curve of degree 3 defined by 8 control points
In fact, there are five Bézier curve segments of degree 3 joining together to form the B-spline curve defined by the control points.

little dots subdivide the B-spline curve into Bézier curve segments.

Subdividing the curve directly is difficult to do so, subdivide the domain of the curve by points called knots.
B-Spline

• In summary, to design a B-spline curve, we need a set of control points, a set of knots and a degree of curve.
B-Spline curve

• \[ P(u) = \sum_{i=0}^{n} N_{i,k}(u)p_i \quad \text{for} \quad (u_0 < u < u_m) \ldots \quad (1.0) \]

Where basis function = \( N_{i,k}(u) \)

Degree of curve \( \rightarrow k-1 \)

Control points, \( p_i \rightarrow 0 < i < n \)

Knot, \( u \rightarrow u_0 < u < u_m \)

\( m = n + k \)
B-Spline : definition

• \( P(u) = \sum N_{i,k}(u)p_i \) \((u_0 < u < u_m)\)
• \( u_i \rightarrow \) knot
• \([u_i, u_{i+1}) \rightarrow \) knot span
• \((u_0, u_1, u_2, \ldots, u_m) \rightarrow \) knot vector
• The point on the curve that corresponds to a knot \( u_i \), \( \rightarrow \) knot point \( P(u_i) \)
• If knots are equally space \( \rightarrow \) uniform (e.g, 0, 0.2, 0.4, 0.6…)
• Otherwise \( \rightarrow \) non uniform (e.g: 0, 0.1, 0.3, 0.4, 0.8 …)
B-Spline : definition

• Uniform knot vector
  – Individual knot value is **evenly spaced**
  – (0, 1, 2, 3, 4)
  – Then, normalized to the range [0, 1]
  – (0, 0.25, 0.5, 0.75, 1.0)
Type of B-Spline uniform knot vector

Non-periodic knots (open knots)
- First and last knots are duplicated k times.
- E.g (0,0,0,1,2,2,2)
- Curve pass through the first and last control points

Periodic knots (non-open knots)
- First and last knots are not duplicated – same contribution.
- E.g (0, 1, 2, 3)
- Curve doesn’t pass through end points.
- used to generate closed curves (when first = last control points)
Type of B-Spline knot vector

- Non-periodic knots (open knots)
- Periodic knots (non-open knots)

(Closed knots)
Non-periodic (open) uniform B-Spline

- The knot spacing is evenly spaced except at the ends where knot values are repeated $k$ times.
- E.g. $P(u) = \sum_{i=0}^{n} N_{i,k}(u)p_i \quad (u_0 < u < u_m)$
- Degree = $k-1$, number of control points = $n + 1$
- Number of knots = $m + 1 @ n + k + 1$
  - for degree = 1 and number of control points = 4 $\rightarrow (k = 2, n = 3)$
  - Number of knots = $n + k + 1 = 6$
- non periodic uniform knot vector (0,0,1,2,3, 3)
  - Knot value between 0 and 3 are equally spaced $\rightarrow$ uniform
Non-periodic (open) uniform B-Spline

- Example
- For curve degree = 3, number of control points = 5
  - $k = 4$, $n = 4$
  - $n + k + 1 = 9$
  - non periodic knots vector = (0,0,0,0,1,2,2,2)
- For curve degree = 1, number of control points = 5
  - $k = 2$, $n = 4$
  - $n + k + 1 = 7$
  - non periodic uniform knots vector = (0, 0, 1, 2, 3, 4, 4, 4)
Non-periodic (open) uniform B-Spline

- For any value of parameters $k$ and $n$, non-periodic knots are determined from

\[
{u_i} = \begin{cases} 
0 & 0 \leq i < k \\
i - k + 1 & k \leq i \leq n \\
- k + 2 & n < i \leq n+k
\end{cases}
\]  

(1.3)

e.g $k=2, n=3$

\[
u_i = \begin{cases} 
0 & 0 \leq i < 2 \\
i - 2 + 1 & 2 \leq i \leq 3 \\
3 - 2 + 2 & 3 < i \leq 5
\end{cases}
\]

\[u = (0, 0, 1, 2, 3, 3)\]
B-Spline basis function

\[ N_{i,k}(u) = (u - u_i) \frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u) \frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}} \]  \hspace{1cm} (1.1)

\[ N_{i,1} = \begin{cases} 
1 & \text{if } u_i \leq u \leq u_{i+1} \\
0 & \text{otherwise} 
\end{cases} \]  \hspace{1cm} (1.2)

→ In equation (1.1), the denominators can have a value of zero, 0/0 is presumed to be zero.

→ If the degree is zero basis function \( N_{i,1}(u) \) is 1 if \( u \) is in the \( i \)-th knot span \([u_i, u_{i+1}]\).
B-Spline basis function

• For example, if we have four knots $u_0 = 0$, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$, knot spans 0, 1 and 2 are $[0,1)$, $[1,2)$, $[2,3)$.

• the basis functions of degree 0 are $N_{0,1}(u) = 1$ on $[0,1)$ and 0 elsewhere, $N_{1,1}(u) = 1$ on $[1,2)$ and 0 elsewhere, and $N_{2,1}(u) = 1$ on $[2,3)$ and 0 elsewhere.

• This is shown below
To understand the way of computing $N_{i,p}(u)$ for $p$ greater than 0, we use the triangular computation scheme.
Non-periodic (open) uniform B-Spline

Example

• Find the knot values of a non periodic uniform B-Spline which has degree $= 2$ and 3 control points. Then, find the equation of B-Spline curve in polynomial form.
Non-periodic (open) uniform B-Spline

Answer

• Degree $= k-1 = 2 \rightarrow k=3$
• Control points $= n + 1 = 3 \rightarrow n=2$
• Number of knot $= n + k + 1 = 6$
• Knot values $\rightarrow u_0=0, u_1=0, u_2=0, u_3=1, u_4=1, u_5= 1$
Non-periodic (open) uniform B-Spline

Answer (cont)

• To obtain the polynomial equation,
  \[ P(u) = \sum_{i=0}^{n} N_{i,k}(u)p_i \]
  
  \[ = \sum_{i=0}^{2} N_{i,3}(u)p_i \]
  
  \[ = N_{0,3}(u)p_0 + N_{1,3}(u)p_1 + N_{2,3}(u)p_2 \]

• firstly, find the \( N_{i,k}(u) \) using the knot value that shown above, start from \( k = 1 \) to \( k=3 \)
Non-periodic (open) uniform B-Spline

Answer (cont)

• For $k = 1$, find $N_{i,1}(u)$ – use equation (1.2):
  
  • $N_{0,1}(u) = \begin{cases} 1 & u_0 \leq u \leq u_1 \text{ ; } (u=0) \\ 0 & \text{otherwise} \end{cases}$

  • $N_{1,1}(u) = \begin{cases} 1 & u_1 \leq u \leq u_2 \text{ ; } (u=0) \\ 0 & \text{otherwise} \end{cases}$

  • $N_{2,1}(u) = \begin{cases} 1 & u_2 \leq u \leq u_3 \text{ ; } (0 \leq u \leq 1) \\ 0 & \text{otherwise} \end{cases}$

  • $N_{3,1}(u) = \begin{cases} 1 & u_3 \leq u \leq u_4 \text{ ; } (u=1) \\ 0 & \text{otherwise} \end{cases}$

  • $N_{4,1}(u) = \begin{cases} 1 & u_4 \leq u \leq u_5 \text{ ; } (u=1) \\ 0 & \text{otherwise} \end{cases}$
Non-periodic (open) uniform B-Spline

Answer (cont)

• For $k = 2$, find $N_{i,2}(u)$ – use equation (1.1):

$$N_{i,k}(u) = (u - u_i) \frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u) \frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}$$

• $N_{0,2}(u) = \frac{u - u_0}{u_1 - u_0} N_{0,1} + \frac{u_2 - u}{u_2 - u_1} N_{1,1} \quad (u_0 = u_1 = u_2 = 0)$

• $u_1 - u_0 \quad u_2 - u_1$

• $= \frac{u - 0}{0 - 0} N_{0,1} + \frac{0 - u}{0 - 0} N_{1,1} = 0$

• $0 - 0 \quad 0 - 0$

• $N_{1,2}(u) = \frac{u - u_1}{u_2 - u_1} N_{1,1} + \frac{u_3 - u}{u_3 - u_2} N_{2,1} \quad (u_1 = u_2 = 0, u_3 = 1)$

• $u_2 - u_1 \quad u_3 - u_2$

• $= \frac{u - 0}{0 - 0} N_{1,1} + \frac{1 - u}{1 - 0} N_{2,1} = 1 - u$

• $0 - 0 \quad 1 - 0$
Non-periodic (open) uniform B-Spline

Answer (cont)

• \( N_{2,2}(u) = \frac{u-u_2}{u_4-u_3} N_{2,1} + \frac{u_4-u}{u_3} N_{3,1} \) \hspace{1cm} (u_2=0, u_3=u_4=1)
  
  • \( u_3-u_2 \quad u_4-u_3 \)
  
  • \( = \frac{u-0}{1-0} N_{2,1} + \frac{1-u}{1-1} N_{3,1} = u \)
  
  • \( 1-0 \quad 1-1 \)

• \( N_{3,2}(u) = \frac{u-u_3}{u_5-u_4} N_{3,1} + \frac{u_5-u}{u_4} N_{4,1} \) \hspace{1cm} (u_3=u_4=u_5=1)
  
  • \( u_4-u_3 \quad u_5-u_4 \)
  
  • \( = \frac{u-1}{1-1} N_{3,1} + \frac{1-u}{1-1} N_{4,1} = 0 \)
  
  • \( 1-1 \quad 1-1 \)
Answer (cont)

For $k = 2$

$N_{0,2}(u) = 0$

$N_{1,2}(u) = 1 - u$

$N_{2,2}(u) = u$

$N_{3,2}(u) = 0$
Non-periodic (open) uniform B-Spline

Answer (cont)

- For \( k = 3 \), find \( N_{i,3}(u) \) – use equation (1.1):

\[
N_{i,k}(u) = (u - u_i) \frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u) \frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}
\]

- \( N_{0,3}(u) = \frac{u - u_0}{u_0 - u_1} N_{0,2} + \frac{u_3 - u}{u_1 - u_2} N_{1,2} \quad (u_0 = u_1 = u_2 = 0, u_3 = 1) \)
- \( = u - 0 \ N_{0,2} + 1 - u \ N_{1,2} = (1-u)(1-u) = (1-u)^2 \)
- \( = 0 - 0 \quad 1 - 0 \)

- \( N_{1,3}(u) = \frac{u - u_1}{u_1 - u_2} N_{1,2} + \frac{u_4 - u}{u_2 - u_3} N_{2,2} \quad (u_1 = u_2 = 0, u_3 = u_4 = 1) \)
- \( = u - 0 \ N_{1,2} + 1 - u \ N_{2,2} = u(1 - u) + (1-u)u = 2u(1-u) \)
- \( = 1 - 0 \quad 1 - 0 \)
Non-periodic (open) uniform B-Spline

Answer (cont)

- \( N_{2,3}(u) = \frac{u - u_2}{u_5 - u_2} N_{2,2} + \frac{u_5 - u}{u_5 - u_3} N_{3,2} \quad (u_2 = 0, u_3 = u_4 = u_5 = 1) \)
- \( \frac{u_4 - u_2}{u_5 - u_3} \)
- \( = \frac{u - 0}{1 - 0} N_{2,2} + \frac{1 - u}{1 - 1} N_{3,2} = u^2 \)
- \( N_{0,3}(u) = (1 - u)^2, \quad N_{1,3}(u) = 2u(1-u), \quad N_{2,3}(u) = u^2 \)

The polynomial equation, \( P(u) = \sum_{i=0}^{n} N_{i,k}(u)p_i \)
- \( P(u) = N_{0,3}(u)p_0 + N_{1,3}(u)p_1 + N_{2,3}(u)p_2 \)
- \( = (1 - u)^2 p_0 + 2u(1-u) p_1 + u^2 p_2 \quad (0 <= u <= 1) \)
Non-periodic (open) uniform B-Spline

- Exercise
- Find the polynomial equation for curve with degree = 1 and number of control points = 4
Non-periodic (open) uniform B-Spline

- Answer
- $k = 2$, $n = 3 \rightarrow$ number of knots = 6
- Knot vector = $(0, 0, 1, 2, 3, 3)$
- For $k = 1$, find $N_{i,1}(u)$ – use equation (1.2):
  - $N_{0,1}(u) = 1 \quad u_0 \leq u \leq u_1 \quad ; \quad (u=0)$
  - $N_{1,1}(u) = 1 \quad u_1 \leq u \leq u_2 \quad ; \quad (0 \leq u \leq 1)$
  - $N_{2,1}(u) = 1 \quad u_2 \leq u \leq u_3 \quad ; \quad (1 \leq u \leq 2)$
  - $N_{3,1}(u) = 1 \quad u_3 \leq u \leq u_4 \quad ; \quad (2 \leq u \leq 3)$
  - $N_{4,1}(u) = 1 \quad u_4 \leq u \leq u_5 \quad ; \quad (u=3)$
Non-periodic (open) uniform B-Spline

Answer (cont)

• For $k = 2$, find $N_{i,2}(u)$ – use equation (1.1):

\[
N_{i,k}(u) = (u - u_i)\frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u)\frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}
\]

• $N_{0,2}(u) = (u - u_0)N_{0,1} + (u_2 - u)N_{1,1}$ (\(u_0 = u_1 = 0, u_2 = 1\))

• \(u_1 - u_0\) \quad \text{and} \quad u_2 - u_1

• \(u - 0\) \quad \text{and} \quad 1 - u

• \(0 - 0\) \quad \text{and} \quad 1 - 0

• \(1 - u\) \quad (0 \leq u \leq 1)
Non-periodic (open) uniform B-Spline

Answer (cont)

- For $k = 2$, find $N_{i,2}(u)$ – use equation (1.1):

\[
N_{i,k}(u) = (u - u_i) \frac{N_{i,k-1}(u)}{u_{i+k-1} - u_i} + (u_{i+k} - u) \frac{N_{i+1,k-1}(u)}{u_{i+k} - u_{i+1}}
\]

- $N_{1,2}(u) = u_2 - u_1 N_{1,1} + u_3 - u N_{2,1}$  \hspace{1cm} (u_1 = 0, u_2 = 1, u_3 = 2)

- $u_2 - u_1 \quad u_3 - u_2$

- $= u - 0 N_{1,1} + 2 - u N_{2,1} \quad 1 - 0 \quad 2 - 1$

- $N_{1,2}(u) = u \quad (0 \leq u \leq 1)$

- $N_{1,2}(u) = 2 - u \quad (1 \leq u \leq 2)$
Non-periodic (open) uniform B-Spline

Answer (cont)

- \( N_{2,2}(u) = \frac{u-u_2}{u_3-u_2} N_{2,1} + \frac{u_4-u}{u_4-u_3} N_{3,1} \) \( (u_2=1, u_3=2,u_4 = 3) \)
- \[ u_3 - u_2 \quad u_4 - u_3 \]
- \[ = \frac{u-1}{2-1} N_{2,1} + \frac{3-u}{3-2} N_{3,1} = \]
- \( N_{2,2}(u) = u - 1 \ (1 \leq u \leq 2) \)
- \( N_{2,2}(u) = 3 - u \ (2 \leq u \leq 3) \)
Non-periodic (open) uniform B-Spline

**Answer (cont)**

- \( N_{3,2}(u) = \frac{u - u_3}{u_4 - u_3} N_{3,1} + \frac{u_5 - u}{u_5 - u_4} N_{4,1} \) \( (u_3 = 2, u_4 = 3, u_5 = 3) \)
- \( u_4 - u_3 \quad u_5 - u_4 \)
- \( = u - 2 \frac{u}{3 - 2} + 3 - u \frac{N_{3,1}}{N_{4,1}} = \)
- \( 3 - 2 \quad 3 - 3 \)
- \( = u - 2 \ (2 \leq u \leq 3) \)
Non-periodic (open) uniform B-Spline

Answer (cont)

- The polynomial equation \( P(u) = \sum N_{i,k}(u)p_i \)
- \( P(u) = N_{0,2}(u)p_0 + N_{1,2}(u)p_1 + N_{2,2}(u)p_2 + N_{3,2}(u)p_3 \)
- \( P(u) = (1 - u) p_0 + u p_1 \) \((0 \leq u \leq 1)\)
- \( P(u) = (2 - u) p_1 + (u - 1) p_2 \) \((1 \leq u \leq 2)\)
- \( P(u) = (3 - u) p_2 + (u - 2) p_3 \) \((2 \leq u \leq 3)\)
Periodic uniform knot

• Periodic knots are determined from
  \[ U_i = i - k \quad (0 \leq i \leq n+k) \]

• Example
  – For curve with degree = 3 and number of control points = 4 (cubic B-spline)
  – \( (k = 4, \ n = 3) \rightarrow \text{number of knots} = 8 \)
  – \( (0, 1, 2, 3, 4, 5, 6, 8) \)
Periodic uniform knot

- Normalize $u$ ($0 \leq u \leq 1$)
- $N_{0,4}(u) = \frac{1}{6} (1-u)^3$
- $N_{1,4}(u) = \frac{1}{6} (3u^3 - 6u^2 + 4)$
- $N_{2,4}(u) = \frac{1}{6} (-3u^3 + 3u^2 + 3u + 1)$
- $N_{3,4}(u) = \frac{1}{6} u^3$

- $P(u) = N_{0,4}(u)p_0 + N_{1,4}(u)p_1 + N_{2,4}(u)p_2 + N_{3,4}(u)p_3$
Periodic uniform knot

- In matrix form
- $P(u) = [u^3, u^2, u, 1].M_n.$
- $M_n = \frac{1}{6} \begin{pmatrix}
-1 & 3 & -3 & 1 \\
3 & -6 & 3 & 0 \\
-3 & 0 & 3 & 0 \\
1 & 4 & 1 & 0
\end{pmatrix}$
Periodic uniform knot

\[ P_0 \]
Closed periodic

Example
$k = 4, n = 5$
Closed periodic

Equation 1.0 change to

\[ N_{i,k}(u) = N_{0,k}((u-i) \text{mod}(n+1)) \]

\[ \Rightarrow P(u) = \sum_{i=0}^{n} N_{0,k}((u-i) \text{mod}(n+1))p_i \]

\[ 0 \leq u \leq n+1 \]
Properties of B-Spline

1. The m degree B-Spline function are piecewise polynomials of degree m \( \Rightarrow \) have \( C^{m-1} \) continuity. \( \Rightarrow \) e.g. B-Spline degree 3 have \( C^2 \) continuity.
Properties of B-Spline

In general, the lower the degree, the closer a B-spline curve follows its control polyline.

Degree = 7  
Degree = 5  
Degree = 3
Properties of B-Spline

Equality $m = n + k$ must be satisfied

Number of knots = $m + 1$

$k$ cannot exceed the number of control points, $n + 1$
Properties of B-Spline

2. Each curve segment is affected by \( k \) control points as shown by past examples. \( \rightarrow \) e.g. \( k = 3 \),
\[
P(u) = N_{i-1,k} p_{i-1} + N_{i,k} p_i + N_{i+1,k} p_{i+1}
\]
Properties of B-Spline

Local Modification Scheme: changing the position of control point $P_i$ only affects the curve $C(u)$ on interval $[u_i, u_{i+k})$.

Modify control point $P_2$
Properties of B-Spline

3. **Strong Convex Hull Property**: A B-spline curve is contained in the convex hull of its control polyline. More specifically, if \( u \) is in knot span \([u_i, u_{i+1})\), then \( C(u) \) is in the convex hull of control points \( P_{i-p}, P_{i-p+1}, \ldots, P_i \).

Degree = 3, \( k = 4 \)
Convex hull based on 4 control points
Properties of B-Spline

4. Non-periodic B-spline curve $C(u)$ passes through the two end control points $P_0$ and $P_n$.

5. Each B-spline function $N_{k,m}(t)$ is nonnegative for every $t$, and the family of such functions sums to unity, that is $\sum_{i=0}^{n} N_{i,k}(u) = 1$

6. Affine Invariance
to transform a B-Spline curve, we simply transform each control points.

7. Bézier Curves Are Special Cases of B-spline Curves
Properties of B-Spline

8. Variation Diminishing: A B-Spline curve does not pass through any line more times than does its control polyline.
Knot Insertion : B-Spline

- *knot insertion* is adding a new knot into the existing knot vector without changing the shape of the curve.
- new knot may be equal to an existing knot $\rightarrow$ the multiplicity of that knot is increased by one
- Since, number of knots $= k + n + 1$
- If the number of knots is increased by 1 $\rightarrow$ either degree or number of control points must also be increased by 1.
- Maintain the curve shape $\rightarrow$ maintain degree $\rightarrow$ change the number of control points.
Knot Insertion : B-Spline

• So, inserting a new knot causes a new control point to be added. In fact, some existing control points are removed and replaced with new ones by corner cutting

Insert knot $u = 0.5$
Single knot insertion : B-Spline

• Given \( n+1 \) control points – \( P_0, P_1, \ldots P_n \)
• Knot vector, \( U = (u_0, u_1, \ldots u_m) \)
• Degree = \( p \), order, \( k = p+1 \)
• Insert a new knot \( t \) into knot vector without changing the shape.
• \( \Rightarrow \) find the knot span that contains the new knot. Let say \( [u_k, u_{k+1}) \)
Single knot insertion : B-Spline

- This insertion will affect to \( k \) (degree + 1) control points (refer to B-Spline properties) \( \rightarrow P_k, P_{k-1}, P_{k-1}, \ldots P_{k-p} \)

- Find \( p \) new control points \( Q_k \) on leg \( P_{k-1}P_k \), \( Q_{k-1} \) on leg \( P_{k-2}P_{k-1} \), \ldots, and \( Q_{k-p+1} \) on leg \( P_{k-p}P_{k-p+1} \) such that the old polyline between \( P_{k-p} \) and \( P_k \) (in black below) is replaced by \( P_{k-p}Q_{k-p+1}Q_kP_k \) (in orange below)
Single knot insertion : B-Spline

• All other control points are not change
• The formula for computing the new control point $Q_i$ on leg $P_{i-1}P_i$ is the following
  • $Q_i = (1-a_i)P_{i-1} + a_i P_i$
  • $a_i = \frac{t-u_i}{u_{i+p} - u_i}$  \hspace{1cm} k-p+1 \leq i \leq k
  • $u_{i+p} - u_i$
Single knot insertion : B-Spline

• Example

• Suppose we have a B-spline curve of degree 3 with a knot vector as follows:

<table>
<thead>
<tr>
<th>$u_0$ to $u_3$</th>
<th>$u_4$</th>
<th>$u_5$</th>
<th>$u_6$</th>
<th>$u_7$</th>
<th>$u_8$ to $u_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.6</td>
<td>0.8</td>
<td>1</td>
</tr>
</tbody>
</table>

Insert a new knot $t = 0.5$ , find new control points and new knot vector?
Single knot insertion: B-Spline

Solution:
- \( t = 0.5 \) lies in knot span \([u_5, u_6)\)
- the affected control points are \( P_5, P_4, P_3 \) and \( P_2 \)
- find the 3 new control points \( Q_5, Q_4, Q_3 \)
- we need to compute \( a_5, a_4 \) and \( a_3 \) as follows

\[
\begin{align*}
- a_5 &= \frac{t - u_5}{u_8 - u_5} = \frac{0.5 - 0.4}{1 - 0.4} = 1/6 \\
- a_4 &= \frac{t - u_4}{u_7 - u_4} = \frac{0.5 - 0.2}{0.8 - 0.2} = 1/2 \\
- a_3 &= \frac{t - u_3}{u_6 - u_3} = \frac{0.5 - 0}{0.6 - 0} = 5/6
\end{align*}
\]
Single knot insertion : B-Spline

• Solution (cont)

• The three new control points are

• $Q_5 = (1-a_5)P_4 + a_5 P_5 = (1-1/6)P_4 + 1/6P_5$

• $Q_4 = (1-a_4)P_3 + a_4 P_4 = (1-1/6)P_3 + 1/6P_4$

• $Q_3 = (1-a_3)P_2 + a_3 P_3 = (1-5/6)P_2 + 5/6P_3$
Single knot insertion : B-Spline

• Solution (cont)
• The new control points are \( P_0, P_1, P_2, Q_3, Q_4, Q_5, P_5, P_6, P_7 \)
• the new knot vector is

<table>
<thead>
<tr>
<th>( u_0 ) to ( u_3 )</th>
<th>( u_4 )</th>
<th>( u_5 )</th>
<th>( u_6 )</th>
<th>( u_7 )</th>
<th>( u_8 )</th>
<th>( u_9 ) to ( u_{12} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
<td>1</td>
</tr>
</tbody>
</table>